Solutions to short-answer questions

1
$$3a + b = 11$$
$$6a + 2b = 22 \quad (1)$$
$$a - 2b = -1 \quad (2)$$
(1) + (2):
$$7a = 21$$
$$a = 3$$
$$3 \times 3 + b = 11$$
$$b = 2$$
$$2 + 2c = 4$$
$$c = 1$$

2
$$\begin{aligned} x^3 &= (x-1)^3 + a(x-1)^2 + b(x-1) + c \\ &= x^3 - 3x^2 + 3x - 1 + ax^2 - 2ax + a + bx - b + c \end{aligned}$$
$$-3 + a = 0$$
$$a = 3$$
$$3 - 2 \times 3 + b = 0$$
$$b = 3$$
$$-1 + 3 - 3 + c = 0$$
$$c = 1$$
$$\therefore \quad x^3 = (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1$$

3
$$\begin{aligned} (x+1)^2(px+q) &= (x^2 + 2x + 1)(px + q) \\ &= px^3 + (q + 2p)x^2 + (p + 2q)x + q \end{aligned}$$
$$a = p$$
$$b = q + 2p$$
$$c = p + 2q$$
$$d = q$$
$$2a + d = 2p + q = b$$
$$a + 2d = p + 2q = c$$

4
$$\begin{aligned} (x-2)^2(px+q) &= (x^2 - 4x + 4)(px + q) \\ &= px^3 + (q - 4p)x^2 + (4p - 4q)x + 4q \end{aligned}$$
$$a = p$$
$$b = q - 4p$$
$$c = 4p - 4q$$
$$d = 4q$$
$$-4a + \frac{1}{4}d = -4p + q = b$$
$$4a - d = 4p - 4q = c$$

5 **a**
$$x^2 + x - 12 = 0$$
$$(x+4)(x-3) = 0$$
$$x = -4 \text{ or } x = 3$$

b
$$x^2 - x - 2 = 0$$
$$(x+1)(x-2) = 0$$
$$x = -1 \text{ or } 2$$

c
$$x^2 - 3x - 11 = -1$$
$$x^2 - 3x - 10 = 0$$
$$(x-5)(x+2) = 0$$
$$x = 5 \text{ or } x = -2$$

d $2x^2 - 4x + 1 = 0$

$$x = \frac{4 \pm \sqrt{16 - 4 \times 2 \times 1}}{4}$$
$$= \frac{4 \pm \sqrt{8}}{4}$$
$$= \frac{2 \pm \sqrt{2}}{2}$$

e $3x^2 - 2x + 5 - t = 0$

$$x = \frac{2 \pm \sqrt{4 - 4 \times 3 \times (5 - t)}}{6}$$
$$= \frac{2 \pm \sqrt{4 - 60 + 12t}}{6}$$
$$= \frac{2 \pm \sqrt{12t - 56}}{6}$$
$$= \frac{2 \pm \sqrt{4(3t - 14)}}{6}$$
$$= \frac{2 \pm 2\sqrt{3t - 14}}{6}$$
$$= \frac{1 \pm \sqrt{3t - 14}}{3}$$

f $tx^2 - tx + 4 = 0$

$$x = \frac{t \pm \sqrt{t^2 - 4 \times t \times 4}}{2t}$$
$$= \frac{t \pm \sqrt{t^2 - 16t}}{2t}$$

6 $$\frac{2(x+2) - 3(x-1)}{(x-1)(x+2)} = \frac{1}{2}$$

$$2(2x + 4 - 3x + 3) = (x - 1)(x + 2)$$
$$2(-x + 7) = x^2 + x - 2$$
$$-2x + 14 = x^2 + x - 2$$
$$x^2 + 3x - 16 = 0$$
$$a = 1,\ b = 3,\ c = -16$$
$$x = \frac{-3 \pm \sqrt{9 - 4 \times 1 \times -16}}{2}$$
$$= \frac{-3 \pm \sqrt{73}}{2}$$

7 a $$\frac{-3x + 4}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$
$$= \frac{A(x+2) + B(x-3)}{(x-3)(x+2)}$$
$$= \frac{Ax + Bx + 2A - 3B}{(x-3)(x+2)}$$

$A + B = -3$

$3A + 3B = -9$ ①

$2A - 3B = 4$ ②

① + ②:

$5A = -5$

$A = -1$

$-1 + B = -3$

$B = -2$

$$\therefore \quad \frac{-3x+4}{(x-3)(x+2)} = -\frac{1}{x-3} - \frac{2}{x+2}$$

b

$$\begin{aligned}\frac{7x+2}{(x+2)(x-2)} &= \frac{A}{x+2} + \frac{B}{x-2} \\ &= \frac{A(x-2)+B(x+2)}{(x+2)(x-2)} \\ &= \frac{Ax+Bx-2A+2B}{(x+2)(x-2)}\end{aligned}$$

$$A+B=7$$

$$2A+2B=14 \quad (1)$$

$$-2A+2B=2 \quad (2)$$

(1) + (2):

$$4B=16$$

$$B=4$$

$$A+4=7$$

$$A=3$$

$$\therefore \quad \frac{7x+2}{(x+2)(x-2)} = \frac{3}{x+2} + \frac{4}{x-2}$$

c

$$\begin{aligned}\frac{7-x}{(x-3)(x+5)} &= \frac{A}{x-3} + \frac{B}{x+5} \\ &= \frac{A(x+5)+B(x-3)}{(x-3)(x+5)} \\ &= \frac{Ax+Bx+5A-3B}{(x-3)(x+5)}\end{aligned}$$

$$A+B=-1$$

$$3A+3B=-3 \quad (1)$$

$$5A-3B=7 \quad (2)$$

(1) + (2):

$$8A=4$$

$$A=\frac{1}{2}$$

$$\frac{1}{2}+B=-1$$

$$B=-\frac{3}{2}$$

$$\therefore \quad \frac{7-x}{(x-3)(x+5)} = \frac{1}{2(x-3)} - \frac{3}{2(x+5)}$$

d

$$\begin{aligned}\frac{3x-9}{(x-5)(x+1)} &= \frac{A}{x-5} + \frac{B}{x+1} \\ &= \frac{A(x+1)+B(x-5)}{(x-5)(x+1)} \\ &= \frac{Ax+Bx+A-5B}{(x-5)(x+1)}\end{aligned}$$

$$A+B=3$$

$$5A+5B=15 \quad (1)$$

$$A-5B=-9 \quad (2)$$

(1) + (2):

$$A = 1$$
$$1 + B = 3$$
$$B = 2$$
$$\therefore \quad \frac{3x-9}{(x-5)(x+1)} = \frac{1}{x-5} + \frac{2}{x+1}$$

e

$$\frac{3x-4}{(x+3)(x+2)^2} = \frac{A}{x+3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$
$$= \frac{A(x+2)^2 + B(x+3)(x+2) + C(x+3)}{(x+3)(x+2)^2}$$
$$= \frac{Ax^2 + 4Ax + 4A + Bx^2 + 5Bx + 6B + Cx + 3C}{(x+3)(x+2)^2}$$

$$A + B = 0$$
$$8A + 8B = 0 \quad (1)$$
$$4A + 5B + C = 3$$
$$12A + 15B + 3C = 9 \quad (2)$$
$$4A + 6B + 3C = -4 \quad (3)$$
(2) − (3):
$$8A + 9B = 13 \quad (4)$$
(4) − (1):
$$B = 13$$
$$A + 13 = 0$$
$$A = -13$$
$$4 \times -13 + 5 \times 13 + C = 3$$
$$C = -10$$
$$\therefore \quad \frac{3x-4}{(x+3)(x+2)^2} = -\frac{13}{x+3} + \frac{13}{x+2} - \frac{10}{(x+2)^2}$$

f

$$\frac{6x^2-5x-16}{(x-1)^2(x+4)} = \frac{A}{x+4} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$
$$= \frac{A(x-1)^2 + B(x+4)(x-1) + C(x+4)}{(x-1)^2(x+4)}$$
$$= \frac{Ax^2 - 2Ax + A + Bx^2 + 3Bx - 4B + Cx + 4C}{(x-1)^2(x+4)}$$

$$A + B = 6$$
$$16A + 16B = 96 \quad (1)$$
$$-2A + 3B + C = -5$$
$$-8A + 12B + 4C = -20 \quad (2)$$
$$A - 4B + 4C = -16 \quad (3)$$
(3) − (2):
$$9A - 16B = 4 \quad (4)$$
(1) + (4):
$$25A = 100$$
$$A = 4$$
$$4 + B = 6$$
$$B = 2$$
$$-2 \times 4 + 3 \times 2 + C = -5$$
$$C = -3$$
$$\therefore \quad \frac{6x^2-5x-16}{(x-1)^2(x+4)} = \frac{4}{x+4} + \frac{2}{x-1} - \frac{3}{(x-1)^2}$$

g

$$\frac{x^2-6x-4}{(x^2+2)(x+1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x+1}$$
$$= \frac{(Ax+B)(x+1)+C(x^2+2)}{(x^2+2)(x+1)}$$
$$= \frac{Ax^2+Ax+Bx+B+Cx^2+2C}{(x^2+2)(x+1)}$$

$A+C=1$ ①

$A+B=-6$ ②

$B+2C=-4$ ③

① − ②:

$C-B=7$ ④

③ + ④:

$3C=3$

$C=1$

$A+1=1$

$A=0$

$0+B=-6$

$B=-6$

$$\therefore \frac{x^2-6x-4}{(x^2+2)(x+1)} = \frac{1}{x+1} - \frac{6}{x^2+2}$$

h

$$\frac{-x+4}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$
$$= \frac{A(x^2+x+1)+(Bx+C)(x-1)}{(x-1)(x^2+x+1)}$$
$$= \frac{Ax^2+Ax+A+Bx^2-Bx+Cx-C}{(x-1)(x^2+x+1)}$$

$A+B=0$ ①

$A-B+C=-1$ ②

$A-C=4$ ③

② + ③:

$2A-B=3$ ④

① + ④:

$3A=3$

$A=1$

$B=-1$

$1-C=4$

$C=-3$

$$\therefore \frac{-x+4}{(x-1)(x^2+x+1)} = \frac{1}{x-1} - \frac{x+3}{x^2+x+1}$$

i

$$\frac{-4x+5}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}$$
$$= \frac{A(x-3)+B(x+4)}{(x+4)(x-3)}$$
$$= \frac{Ax+Bx-3A+4B}{(x+4)(x-3)}$$

$A+B=-4$

$$3A + 3B = -12 \quad ①$$
$$-3A + 4B = 5 \quad ②$$
$$① + ②: 7B = 7$$
$$B = -1$$
$$A - 1 = -4$$
$$A = -3$$
$$\therefore \frac{-4x+5}{(x+4)(x-3)} = -\frac{3}{x+4} - \frac{1}{x-3}$$
$$= \frac{1}{3-x} - \frac{3}{x+4}$$

j

$$\frac{-2x+8}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}$$
$$= \frac{A(x-3)+B(x+4)}{(x+4)(x-3)}$$
$$= \frac{Ax+Bx-3A+4B}{(x+4)(x-3)}$$
$$A + B = -2$$
$$3A + 3B = -6 \quad ①$$
$$-3A + 4B = 8 \quad ②$$
$$① + ②: 7B = 2$$
$$B = \frac{2}{7}$$
$$A + \frac{2}{7} = -2$$
$$A = -\frac{16}{7}$$
$$\therefore \frac{-2x+8}{(x+4)(x-3)} = \frac{2}{7(x-3)} - \frac{16}{7(x+4)}$$

8 a

$$\frac{14x-28}{(x-3)(x^2+x+2)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+x+2}$$
$$= \frac{A(x^2+x+2)+(Bx+C)(x-3)}{(x-3)(x^2+x+2)}$$
$$= \frac{Ax^2+Ax+2A+Bx^2-3Bx+Cx-3C}{(x-3)(x^2+x+2)}$$
$$A + B = 0$$
$$9A + 9B = 0 \quad ①$$
$$A - 3B + C = 14$$
$$3A - 9B + 3C = 42 \quad ②$$
$$2A - 3C = -28 \quad ③$$
$$② + ③: 5A - 9B = 14 \quad ④$$
$$① + ④: 14A = 14$$
$$A = 1$$
$$1 + B = 0$$
$$B = -1$$
$$1 - 3 \times -3 + C = 14$$
$$C = 10$$
$$\therefore \frac{14x-28}{(x-3)(x^2+x+2)} = \frac{1}{x-3} + \frac{-x+10}{x^2+x+2}$$
$$= \frac{1}{x-3} - \frac{x-10}{x^2+x+2}$$

b $$\frac{1}{(x+1)(x^2-x+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+2}$$
$$= \frac{A(x^2-x+2)+(Bx+C)(x+1)}{(x+1)(x^2-x+2)}$$
$$= \frac{Ax^2-Ax+2A+Bx^2+Bx+Cx+C}{(x+1)(x^2-x+2)}$$

$$A+B=0 \quad (1)$$
$$-A+B+C=0 \quad (2)$$
$$2A+C=1 \quad (3)$$
$$(3)-(2): 3A-B=1 \quad (4)$$
$$(1)+(4): 4A=1$$
$$A=\frac{1}{4}$$
$$\frac{1}{4}+B=0$$
$$B=-\frac{1}{4}$$
$$-\frac{1}{4}-\frac{1}{4}+C=0$$
$$C=\frac{1}{2}$$
$$\therefore \frac{1}{(x+1)(x^2-x+2)} = \frac{1}{4(x+1)} + \frac{-x+2}{4(x^2-x+2)}$$
$$= \frac{1}{4(x+1)} - \frac{x-2}{4(x^2-x+2)}$$

c First divide $3x^3$ by x^2-5x+4.

$$\begin{array}{r} 3x+15 \\ x^2-5x+4\,\overline{)\,3x^3} \\ 3x^3-15x^2+12x \\ \hline 15x^2-12x \\ 15x^2-75x+60 \\ \hline 63x-60 \end{array}$$

$$\frac{3x^3}{x^2-5x+4} = 3x+15+\frac{63x-60}{(x-4)(x-1)}$$ (factorising the denominator)

$$\frac{63x-60}{(x-4)(x-1)} = \frac{A}{x-4}+\frac{B}{x-1}$$
$$= \frac{A(x-1)+B(x-4)}{(x-4)(x-1)}$$
$$= \frac{Ax+Bx-A-4B}{(x-4)(x-1)}$$

$$A+B=63 \quad (1)$$
$$-A-4B=-60 \quad (2)$$
$$(1)+(2): -3B=3$$
$$B=-1$$
$$A-1=63$$
$$A=64$$
$$\therefore \frac{63x-60}{(x-4)(x-1)} = \frac{64}{x-4}-\frac{1}{x-1}$$
$$\frac{3x^3}{x^2-5x+4} = 3x+15+\frac{64}{x-4}-\frac{1}{x-1}$$

9 a

$$\begin{aligned} x^2 &= -x \\ x^2 + x &= 0 \\ x(x+1) &= 0 \\ x &= 0 \text{ or } x = -1 \end{aligned}$$

If $x = 0,\ y = 0$

If $x = -1,\ y = 1$

The points of intersection are $(0,0)$ and $(-1,1)$.

b Substitute $y = 4 - x$ into $x^2 + y^2 = 16$.

$$\begin{aligned} x^2 + (4-x)^2 &= 16 \\ x^2 + 16 - 8x + x^2 &= 16 \\ 2x^2 - 8x &= 0 \\ x^2 - 4x &= 0 \\ x(x-4) &= 0 \\ x &= 0 \text{ or } x = 4 \end{aligned}$$

If $x = 0,\ y = 4$

If $x = 4,\ y = 0$

The points of intersection are $(0,4)$ and $(4,\ 0)$.

c Substitute $y = 5 - x$ into $xy = 4$.

$$\begin{aligned} x(5-x) &= 4 \\ 5x - x^2 - 4 &= 0 \\ x^2 - 5x + 4 &= 0 \\ (x-4)(x-1) &= 0 \\ x = 4 &\text{ or } x = 1 \end{aligned}$$

If $x = 4,\ y = 1$

If $x = 1,\ y = 4$

The points of intersection are $(4,1)$ and $(1,4)$.

10 Substitute $x = 3y - 1$ into the circle.

$$\begin{aligned} (3y-1)^2 + 2(3y-1) + y^2 &= 9 \\ 9y^2 - 6y + 1 + 6y - 2 + y^2 &= 9 \\ 10y^2 - 10 &= 0 \\ y^2 - 1 &= 0 \\ (y+1)(y-1) &= 0 \\ y &= 1 \text{ or } y = -1 \end{aligned}$$

If $y = -1,\ x = -4$

If $y = 1,\ x = 2$

The points of intersection are $(2,1)$ and $(-4,-1)$.

11a $t = \dfrac{135}{x}$

b $t = \dfrac{135}{x - 15}$

c $x = 60$

d $60\ \text{km/h}, 45\ \text{km/h}$

Solutions to multiple-choice questions

1 **C** $x^2 = (x+1)^2 + b(x+1) + c$

$$= x^2 + 2x + 1 + bx + b + c$$

$$\begin{aligned} b + 2 &= 0 \\ b &= -2 \\ b + c + 1 &= 0 \\ c &= 1 \end{aligned}$$

2 **D** $x^3 = a(x+2)^3 + b(x+2)^2 + c(x+2) + d$

$$= ax^3 + 6ax^2 + 12ax + 8a + bx^2 + 4bx + 4b + cx + 2c + d$$

$$\begin{aligned} a &= 1 \\ b + 6a &= 0 \\ b &= -6 \\ 12a + 4b + c &= 0 \\ c &= 12 \\ 8a + 4b + 2c + d &= 0 \\ d &= -8 \end{aligned}$$

3 **D** $a = 3,\ b = -6,\ c = 3$

$$\begin{aligned} x &= \frac{6 \pm \sqrt{36 - 4 \times 3 \times 3}}{2 \times 3} \\ &= \frac{6 \pm \sqrt{0}}{6} \\ &= 1 \end{aligned}$$

4 **C** $(x-4)(x+6) = 0$

$$\begin{aligned} x^2 + 2x - 24 &= 0 \\ x^2 + 2x &= 24 \\ 2x^2 + 4x &= 48 \end{aligned}$$

5 **E**

$$\begin{aligned} \frac{3}{x+4} - \frac{5}{x-2} &= \frac{3(x-2) - 5(x+4)}{(x+4)(x-2)} \\ &= \frac{3x - 6 - 5x - 20}{(x+4)(x-2)} \\ &= \frac{-2x - 26}{(x+4)(x-2)} \\ &= \frac{-2(x+13)}{(x+4)(x-2)} \end{aligned}$$

6 **E**

$$\begin{aligned} \frac{4}{(x+3)^2} + \frac{2x}{x+1} &= \frac{4(x+1) + 2x(x+3)^2}{(x+3)^2(x+1)} \\ &= \frac{4x + 4 + 2x^3 + 12x^2 + 18x}{(x+3)^2(x+1)} \\ &= \frac{2x^3 + 12x + 22x + 4}{(x+3)^2(x+1)} \\ &= \frac{2(x^3 + 6x^2 + 11x + 2)}{(x+3)^2(x+1)} \end{aligned}$$

7 **C**

$$\frac{7x^2+13}{(x-1)(x^2+x+2)} = \frac{a}{x-1} + \frac{bx+c}{x^2+x+2}$$
$$= \frac{a(x^2+x+2)+(bx+c)(x-1)}{(x-1)(x^2+x+2)}$$
$$= \frac{ax^2+ax+2a+bx^2-bx+cx-c}{(x-1)(x^2+x+2)}$$

$a+b=7$ ①

$a-b+c=0$ ②

$2a-c=13$ ③

② + ③:

$3a-b=13$ ④

① + ④:

$4a=20$

$a=5$

$5+b=7$

$b=2$

$a-b+c=0$

$C=-3$

8 **D**

$$\frac{4x-3}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$
$$= \frac{A(x-3)+B}{(x-3)^2}$$
$$= \frac{Ax-3A+B}{(x-3)^2}$$

$A=4$

$-3\times 4+B=-3$

$B=9$

$$\therefore \frac{4x-3}{(x-3)^2} = \frac{4}{x-3} + \frac{9}{(x-3)^2}$$

9 **B** $2x^2+5x+2=(2x+1)(x+2)$

$$\frac{8x+7}{(2x+1)(x+2)} = \frac{A}{2x+1} + \frac{B}{x+2}$$
$$= \frac{A(x+2)+B(2x+1)}{(2x+1)(x+2)}$$
$$= \frac{Ax+2Bx+2A+B}{(2x+1)(x+2)}$$

$A+2B=8$

$2A+4B=16$ ①

$2A+B=7$ ②

① − ②:

$3B=9$

$B=3$

$A+2B=8$

$A=2$

$$\therefore \frac{8x+7}{(2x+1)(x+2)} = \frac{2}{2x+1} + \frac{3}{x+2}$$

10 **B** $\dfrac{-3x^2+2x-1}{(x^2+1)(x+1)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x+1}$

$$= \frac{(Ax+B)(x+1)+C(x^2+1)}{(x^2+1)(x+1)}$$

$$= \frac{Ax^2+Ax+Bx+B+Cx^2+C}{(x^2+1)(x+1)}$$

$$\begin{aligned} A+C &= -3 \quad ① \\ A+B &= 2 \quad ② \\ B+C &= -1 \quad ③ \end{aligned}$$

① − ②:

$$\begin{aligned} C-B &= -5 \\ 2C &= -6 \\ C &= -3 \\ A+-3 &= -3 \\ A &= 0 \\ 0+B &= 2 \\ B &= 2 \end{aligned}$$

$$\therefore \quad \frac{-3x+2x+5}{(x^2+1)(x+1)} = \frac{2}{x^2+1} - \frac{3}{x+1}$$

Solutions to extended-response questions

1 **a** Let V km/h be the initial speed.

$V-4$ is the new speed.

It takes 2 more hours to travel at the new speed,

$$\begin{aligned} \therefore \quad \frac{240}{V}+2 &= \frac{240}{V-4} \quad \ldots \boxed{1} \\ \therefore \quad 240(V-4)+2V(V-4) &= 240V \\ \therefore \quad 240V-960+2V^2-8V &= 240V \\ \therefore \quad 2V^2-8V-960 &= 0 \\ \therefore \quad V^2-4V-480 &= 0 \\ \therefore \quad (V-24)(V+20) &= 0 \\ \therefore \quad V &= 24 \text{ or } V=-20 \end{aligned}$$

Actual speed is 24 km/h.

b If it travels at $V-a$ km/h and takes 2 more hours, equation $\boxed{1}$ from **a** becomes

$$\begin{aligned} \frac{240}{V}+2 &= \frac{240}{V-a} \\ \therefore \quad 240(V-a)+2V(V-a) &= 240V \\ \therefore \quad 240V-240a+2V^2-2Va &= 240V \\ \therefore \quad 2V^2-2aV-240a &= 0 \\ \therefore \quad V^2-aV-120a &= 0 \end{aligned}$$

Using the general quadratic formula,

$$V = \frac{a+\sqrt{a^2+480a}}{2}$$

When $a=60,\ V=120$, i.e. the speed is 120 km/h, a fairly fast speed. So if speed is less than this, practical values are $0<a<60$ and then $0<V<120$.

c If it travels at $V-a$ km/h and takes a more hours, equation $\boxed{1}$ from **a** becomes

$$\frac{240}{V}+a=\frac{240}{V-a}$$

$$\begin{aligned}
\therefore \quad 240(V-a)+aV(V-a)&=240V\\
\therefore \quad 240V-240a+aV^2-a^2V&=240V\\
\therefore \quad aV^2-a^2V-240a&=0\\
\therefore \quad V^2-aV-240&=0
\end{aligned}$$

Using the general quadratic formula,

$$V=\frac{a+\sqrt{a^2+960}}{2}$$

The only pairs of integers for a and V are found in the table below.

a	1	8	14	22	34	43	56	77	118
V	16	20	24	30	40	48	60	80	120

2 A table is a useful way to display the speed, time taken and distance covered for each train.

	distance (km)	time (h)	speed (km/h)
Faster train	b	$\frac{b}{v}$	v
Slower train	b	$\frac{b}{v}+a$	$b\div\left(\frac{b}{v}+a\right)=\frac{bv}{b+av}$

a In c hours, the faster train travels a distance of cv km.

In c hours, the slower train travels a distance of $\frac{bcv}{b+av}$ km.

Since the slower train travels 1 km less than the faster one in c hours,

$$\begin{aligned}
cv-1&=\frac{bcv}{b+av}\\
\therefore (cv-1)(b+av)&=bcv\\
\therefore bcv+acv^2-b-av&=bcv\\
\therefore acv^2-av-b&=0
\end{aligned}$$

Using the general quadratic formula,

$$\begin{aligned}
v&=\frac{a\pm\sqrt{a^2+4abc}}{2ac}\\
&=\frac{a+\sqrt{a^2+4abc}}{2ac}\text{ since } v>0
\end{aligned}$$

Therefore the speed of the faster train is $\frac{a+\sqrt{a^2+4abc}}{2ac}$ km/h.

b If the speed of the faster train is a rational number, then a^2+4abc must be a square number.

Set 1

If $a=1$,

then $a^2+4abc=1+4bc$

e.g. $a=1,\ b=1,\ c=2$

$$\text{in which case } v = \frac{a+\sqrt{a^2+4abc}}{2ac}$$

$$\text{becomes } v = \frac{1+\sqrt{1^2+4\times1\times1\times2}}{2\times1\times2}$$

$$= \frac{1+\sqrt{9}}{4}$$

$$= 1\ \mathrm{km/h}$$

Set 2

If $a = 1$ and $b = 100$,

then $a^2 + 4abc = 1 + 400c$

Choose $c = \frac{11}{10}$

$$\text{then } a^2 + 4ac = 1 + 400 \times \frac{11}{10}$$

$$= 441$$

$$= 21^2$$

When $a = 1,\ b = 100$ and $c = \frac{11}{10}$,

$$v = \frac{a+\sqrt{a^2+4abc}}{2ac}$$

$$\text{becomes } v = \frac{1+21}{2\times1\times\frac{11}{10}}$$

$$= \frac{22\times10}{22}$$

$$= 10\ \mathrm{km/h}$$

Set 3

If $a = \frac{1}{2},\ b = 15,\ c = 1$

$$\text{then } v = \frac{a+\sqrt{a^2+4abc}}{2ac}$$

$$\text{becomes } v = \frac{\frac{1}{2}+\sqrt{\left(\frac{1}{2}\right)^2+4\times\frac{1}{2}\times15\times1}}{2\times\frac{1}{2}\times1}$$

$$= \frac{\frac{1}{2}+\sqrt{\frac{121}{4}}}{1}$$

$$= \frac{1}{2}+\frac{11}{2}$$

$$= 6\ \mathrm{km/h}$$

Set 4

If $a = \frac{1}{4}$,

then $a^2 + 4abc = \frac{1}{16} + bc$

e.g. $a = 1,\ b = 5,\ c = 1$

$$\text{in which case } v = \frac{a+\sqrt{a^2+4abc}}{2ac}$$

$$\text{becomes } v = \frac{\frac{1}{4}+\sqrt{\left(\frac{1}{4}\right)^2+4\times\frac{1}{4}\times 5\times 1}}{2\times 1\times 1}$$

$$= \frac{\frac{1}{4}+\sqrt{\frac{81}{16}}}{2}$$

$$= \frac{5}{4}\ \mathrm{km/h}$$

Set 5

If $a=1$ and $b=1$,

then $a^2+4abc=1+4c$

Choose $c=6$

$$\begin{aligned}\text{then } a^2+4ac &= 1+4\times 6\\ &= 25\\ &= 5^2\end{aligned}$$

When $a=1,\ b=1$ and $c=6$,

$$v = \frac{a+\sqrt{a^2+4abc}}{2ac}$$

$$\text{becomes } v = \frac{1+\sqrt{1^2+4\times 1\times 1\times 6}}{2\times 1\times 6}$$

$$= \frac{1+5}{12}$$

$$= \frac{1}{2}\ \mathrm{km/h}$$

3 a

	Volume	Time	Rate
Large pipe	1	T_L	r_L
Small pipe	1	T_S	r_S
Both pipes	1	T_B	r_L+r_S

T_L is the time for the large pipe to fill the tank

T_S is the time for the small pipe to fill the tank

T_B is the time for both pipes to fill the tank

where it is assumed without loss of generality that the volume of the tank is 1 unit.

Given

$$T_S = T_L + a \qquad \ldots \boxed{1}$$
$$T_S = T_B + b \qquad \ldots \boxed{2}$$

Note that $r_B = r_S + r_L$.

$$T_B = \frac{1}{r_B} = \frac{1}{r_S + r_L} = \frac{1}{\frac{1}{T_S} + \frac{1}{T_L}} = \frac{T_S T_L}{T_S + T_L}$$

From $\boxed{1}$ and $\boxed{2}$

$$T_L + a = T_B + b = \frac{T_S T_L}{T_S + T_L} + b$$

$$\therefore T_L(T_L + T_S) + a(T_L + T_S) = T_S T_L + b(T_L + T_S)$$
$$\therefore T_L(2T_L + a) + a(2T_L + a) = T_L(T_L + a) + b(2T_L + a)$$
$$\therefore 2T_L^2 + aT_L + 2aT_L + a^2 = T_L^2 + aT_L + 2bT_L + ba$$
$$\therefore T_L^2 + 2(a-b)T_L + a^2 - ba = 0$$

$$\therefore \quad T_L = \frac{2(b-a) + \sqrt{4(a^2 - 2ab + b^2) - 4(a^2 - ba)}}{2} \text{ since } T_L > 0$$
$$= \frac{2(b-a) + \sqrt{4a^2 - 8ab + 4b^2 - 4a^2 + 4ba}}{2}$$
$$= b - a + \sqrt{-ab + b^2}$$

Also from $\boxed{1}$ $T_S = T_L + a$
$$= b - a + \sqrt{b^2 - ab} + a$$
$$= b + \sqrt{b^2 - ab}$$

b If $a = 24$ and $b = 32$,

$$T_S = 32 + \sqrt{32^2 - 32 \times 24} = 48$$
$$T_L = T_S - a = 48 - 24 = 24$$

c $b^2 - ab$ is a perfect square, and $T_S = b + \sqrt{b^2 - ab}$.

Let $b = a + 1$. Then $T_S = a + 1 + \sqrt{(a+1)^2 - a(a+1)}$
$$= a + 1 + \sqrt{a^2 + 2a + 1 - a^2 - a}$$
$$= a + 1 + \sqrt{a + 1}$$

Note: This means b must be a perfect square.

a	3	8	15	24	35
b	4	9	16	25	36
T_S	8	18	32	50	72
T_L	5	10	17	26	37